Hicksian, expenditure function, indirect utility and Roy's lemma for cobb-douglas

Consider the case of a consumer with preferences over two goods x and y, represented by the following utility function $U(x_1, x_2) = x_1^a x_2^b$.

1. Find the Hicksian demand function for each good.
2. What is the problem that originates them? What is the minimum expenditure function and how is it defined?
3. From the previous point, find the indirect utility function. What does it indicate?
4. Find the Marshallian demand functions from the minimum expenditure function. Economically interpret both functions.

Solution

1. We set up the Lagrangian associated with the utility minimization problem:

$$L = p_1x_1 + p_2x_2 + \lambda(\bar{U} - x_1^a x_2^b)$$

$$L'x_1 = p_1 - \lambda a x_1^{a-1} x_2^b = 0$$

$$L'x_2 = p_2 - \lambda b x_1^a x_2^{b-1} = 0$$

$$L'\lambda = \bar{U} - x_1^a x_2^b = 0$$

From the first two equations:

$$\frac{p_1}{a x_1^{a-1} x_2^b} = \lambda$$

$$\frac{p_2}{b x_1^a x_2^{b-1}} = \lambda$$

Equalize the λ:

$$\frac{p_1}{a x_1^{a-1} x_2^b} = \frac{p_2}{b x_1^a x_2^{b-1}}$$

$$\frac{x_1 p_1}{a} = \frac{p_2 x_2}{b}$$

$$x_1 = \frac{a p_2 x_2}{p_1 b}$$

Insert into the budget constraint:

$$\bar{U} - [\frac{a p_2 x_2}{p_1 b}]^a x_2^b = 0$$

$$\bar{U} = [\frac{a p_2}{p_1 b}]^a x_2^{a+b}$$

$$\bar{U}[\frac{b p_1}{a p_2}]^a = x_2^{a+b}$$

$$x_2 = (\bar{U}[\frac{b p_1}{a p_2}]^a)^{1/(a+b)}$$

$$h_2 = \bar{U}^{1/(a+b)}[\frac{b p_1}{a p_2}]^{a/(a+b)}$$

Insert into x_1:

$$x_1 = \frac{a p_2}{p_1 b}\bar{U}^{1/(a+b)}[\frac{b p_1}{a p_2}]^{a/(a+b)}$$

$$x_1 = \bar{U}^{1/(a+b)}[\frac{b p_1}{a p_2}]^{a/(a+b)-1} = \bar{U}^{1/(a+b)}[\frac{b p_1}{a p_2}]^{-b/(a+b)}$$

$$h_1^m = \bar{U}^{1/(a+b)}[\frac{a p_2}{b p_1}]^{b/(a+b)}$$

2. The Hicksian demands or compensated demands for goods x_1 and x_2 arise from the problem of minimizing the total necessary expenditure to reach a given utility level ($\bar{U}$), keeping this utility level constant despite variations in prices. This problem is formalized as:

$$\min_{x_1,x_2} p_1x_1 + p_2x_2$$

subject to the constraint that the utility reached is at least as great as a predefined utility level $\bar{U}$, that is,

$$U(x_1, x_2) = x_1^a x_2^b \geq \bar{U}.$$

Minimum Expenditure Function

The minimum expenditure function, denoted as $e(p_1, p_2, \bar{U})$, is the solution to the aforementioned expenditure minimization problem. It is defined as the minimum expenditure necessary to reach the utility level $\bar{u}$ given the prices p_1 and p_2 of goods x_1 and x_2. The minimum expenditure function is mathematically expressed as:

$$e(p_1, p_2, \bar{U}) = \min_{x_1, x_2} \{p_1 x_1 + p_2 x_2 \mid x_1^a x_2^b \geq \bar{U}\}.$$

$$e(p_1, p_2, \bar{U}) = p_1 x_1^h + p_2 x_2^h$$

$$e(p_1, p_2, \bar{U}) = p_1 \bar{U}^{1/(a+b)} [\frac{ap_2}{bp_1}]^{b/(a+b)} + p_2 \bar{U}^{1/(a+b)} [\frac{bp_1}{ap_2}]^{a/(a+b)}$$

$$e(p_1, p_2, \bar{U}) = \bar{U}^{1/(a+b)} [p_1 [\frac{ap_2}{bp_1}]^{b/(a+b)} + p_2 [\frac{bp_1}{ap_2}]^{a/(a+b)}]$$

$$e(p_1, p_2, \bar{U}) = \bar{U}^{1/(a+b)} [p_1^{a/(a+b)} [\frac{ap_2}{b}]^{b/(a+b)} + p_2^{b/(a+b)} [\frac{bp_1}{a}]^{a/(a+b)}]$$

$$e(p_1, p_2, \bar{U}) = \bar{U}^{1/(a+b)} p_1^{a/(a+b)} p_2^{b/(a+b)} [[\frac{a}{b}]^{b/(a+b)} + [\frac{b}{a}]^{a/(a+b)}]$$

$$e(p_1, p_2, \bar{U}) = \bar{U}^{1/(a+b)} p_1^{a/(a+b)} p_2^{b/(a+b)} [\frac{a^{b/(a+b)} a^{a/(a+b)} + b^{b/(a+b)} b^{a/(a+b)}}{b^{b/(a+b)} a^{a/(a+b)}}]$$

$$e(p_1, p_2, \bar{U}) = \bar{U}^{1/(a+b)} p_1^{a/(a+b)} p_2^{b/(a+b)} [\frac{a+b}{b^{b/(a+b)} a^{a/(a+b)}}]$$

3. The indirect utility function $v(p_1, p_2, M)$ is defined as the maximum utility level that a consumer can achieve given a set of prices (p_1, p_2) and an income level M. Formally, it is expressed as:

$$v(p_1, p_2, M) = \max_{x_1, x_2} \{U(x_1, x_2) \mid p_1 x_1 + p_2 x_2 \leq M\}$$

This function arises from inverting the minimum expenditure function, which seeks to determine the least expenditure necessary to reach a certain utility level. Mathematically, the minimum expenditure function $e(p_1, p_2, u)$ is:

$$e(p_1, p_2, u) = \min_{x_1, x_2} \{p_1 x_1 + p_2 x_2 \mid U(x_1, x_2) \geq u\}$$

The duality between these functions is revealed by considering that the indirect utility v and the expenditure function e are inverses of each other in the sense that they complement each other to connect the utility level and the associated expenditure, under constraints of prices and income.

Inverting:

$$v(p_1, p_2, M) = M^{a+b} \left(\frac{b^{b/(a+b)} a^{a/(a+b)}}{(a+b) p_1^{a/(a+b)} p_2^{b/(a+b)}} \right)^{a+b}$$

$$v(p_1, p_2, M) = \frac{M^{a+b}}{p_1^a p_2^b} \left(\frac{b^b a^a}{(a+b)^{a+b}} \right)^{a+b}$$

4. Before working with $v(p_1, p_2, M)$, we express it in another way to make it simpler to derive:

$$v(p_1, p_2, M) = M^{a+b} p_1^{-a} p_2^{-b} \left(\frac{b^b a^a}{(a+b)^{a+b}} \right)^{a+b}$$

Derive the indirect utility function:

$$\frac{\partial v}{\partial p_1} = -aM^{a+b} p_1^{-a-1} p_2^{-b} \left(\frac{b^b a^a}{(a+b)^{a+b}} \right)^{a+b}$$

$$\frac{\partial v}{\partial p_2} = -bM^{a+b} p_1^{-a} p_2^{-b-1} \left(\frac{b^b a^a}{(a+b)^{a+b}} \right)^{a+b}$$

$$\frac{\partial v}{\partial M} = (a+b)M^{a+b-1} p_1^{-a} p_2^{-b} \left(\frac{b^b a^a}{(a+b)^{a+b}} \right)^{a+b}$$

By Roy's lemma:

$$\frac{v' p_1}{v' m} = \frac{-aM^{a+b} p_1^{-a-1} p_2^{-b} \left(\frac{b^b a^a}{(a+b)^{a+b}} \right)^{a+b}}{(a+b)M^{a+b-1} p_1^{-a} p_2^{-b} \left(\frac{b^b a^a}{(a+b)^{a+b}} \right)^{a+b}}$$

$$\frac{v' p_1}{v' m} = \frac{-aM}{(a+b)p_1} = -x_1^m$$

$$\frac{v' p_2}{v' m} = \frac{-bM^{a+b} p_1^{-a} p_2^{-b-1} \left(\frac{b^b a^a}{(a+b)^{a+b}} \right)^{a+b}}{(a+b)M^{a+b-1} p_1^{-a} p_2^{-b} \left(\frac{b^b a^a}{(a+b)^{a+b}} \right)^{a+b}}$$

$$\frac{v' p_1}{v' m} = \frac{-bM}{(a+b)p_1} = -x_2^m$$

The Marshallian demands represent the quantities of goods and services that a consumer chooses to maximize their utility, given their income and the prices of goods. Roy's lemma underscores how a consumer optimally reacts to changes in prices and income to maximize their utility within budget constraints. The consumer will adjust their purchases such that the ratio of the change in marginal utility per good (due to a change in its price) is proportional to the change in marginal utility of income.